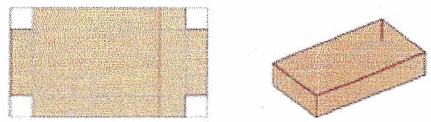
1. The product of two positive numbers is 363. Minimize the sum of the first and three times the second. (answer: 33 and 11)

2. The management of a large store wishes to add a fenced-in rectangular storage yard of 20,000 sq. ft., using the building as one side of the yard. Find the minimum amount of fencing that must be used to enclose the remaining 3 sides of the yard. (answer: 400')

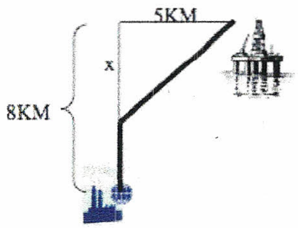
3. A farmer has 80 feet of fencing to enclose three sides of a rectangular plot of land (the fourth side has an existing stone partition). What dimensions for the rectangular plot will enclose the maximum area? (answer: 20'x40')

4. An open box is to be made from a rectangular piece of material by cutting equal squares from each corner and turning up the sides. Find the dimensions of the box of maximum volume if the material has dimensions 6 inches by 6 inches. (answer: 4" x 4" x 1")



5. An open box is to be made from a 16" by 30" piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with largest possible volume? (Answer: 10/3")

6. An offshore oil well is located in the ocean at point W, which is 5 miles from the closest shorepoint A an a straight shoreline. The oil is to be piped to a shorepoint B that is 8 miles from A by piping it on a straight line under water from W to some shorepoint P between A and B and then on to B via a pipe along the shoreline. If the cost of laying the pipe is \$100,000 per mile under water and \$75,000 per mile over land, where should the point P be located to minimize the cost of laying the pipe? (Answer: approx 5.67 miles from A)

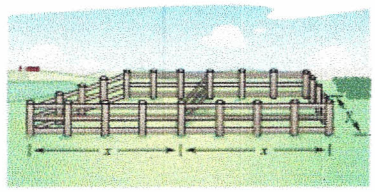


7. The sum of the first number and twice the second number is 108 and the product is a maximum. Find two positive numbers that satisfy this requirement. (Answer: 27 and 54)

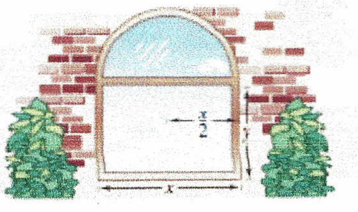
8. Find the point on the graph of the function that is closest to the given point:

$$f(x) = x^2, \quad \left(2, \frac{1}{2}\right) \qquad \text{(Answer: } (1,1)\text{)}$$

9. A rancher has 400' of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum? (Answer: 50' x 200/3')



10. A Norman window is constructed by adjoining a semicircle on the top of an ordinary rectangular window. Find the dimensions of the Norman window of maximum are if the total perimeter is 16 feet. (Answer: $\frac{32}{4+\pi}$' x $\frac{16}{4+\pi}$')



MAC 2311 OPTIMIZATION PROBLEMS → SOLUTIONS

#1

PRIMARY EQUATION (MIN, SUM)

SECONDARY EQUATION

$$xy = 363$$

$$y = \frac{363}{x}$$

$$S = x + 3y$$

$$S = x + 3\left(\frac{363}{x}\right)$$

$$S = x + 1089x^{-1}$$

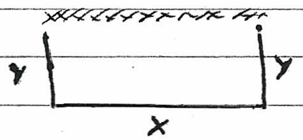
$$S' = 1 - \frac{1089}{x^2}$$

$$1 - \frac{1089}{x^2} = 0$$

$$x^2 = 1089$$

$$x = 33 \Rightarrow y = \frac{363}{33} \Rightarrow y = 11$$

#2



PRIMARY EQUATION (MIN. PERIMETER)

SECONDARY EQUATION

$$20{,}000 = xy$$

$$y = \frac{20{,}000}{x}$$

$$P = x + 2y$$

$$P = x + 2\left(\frac{20{,}000}{x}\right)$$

$$P = x + 40{,}000x^{-1}$$

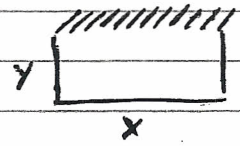
$$P' = 1 - \frac{40{,}000}{x^2}$$

$$1 - \frac{40{,}000}{x^2} = 0$$

$$x = 200' \Rightarrow y = \frac{20{,}000}{200} \Rightarrow y = 100'$$

MINIMUM AMOUNT OF FENCING NEEDED IS $200' + 100' + 100' = 400'$

#3



PRIMARY EQUATION (MAX. AREA)

$$A = xy$$

SECONDARY EQUATION

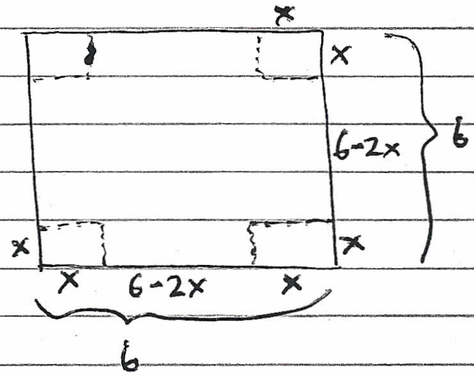
$$80 = 2y + x$$
$$x = 80 - 2y$$

$$A = (80-2y)y$$
$$A = 80y - 2y^2$$

$$A' = 80 - 4y$$
$$80 - 4y = 0$$

$$y = 20' \Rightarrow x = 80 - 2(20) \Rightarrow x = 40'$$

#4



PRIMARY EQUATION (MAX VOLUME)

$$V = lwh$$
$$V = (6-2x)(6-2x)(x)$$
$$V = 36x - 24x^2 + 4x^3$$
$$V' = 36 - 48x + 12x^2$$

$$12x^2 - 48x + 36 = 0$$
$$12(x^2 - 4x + 3) = 0$$
$$12(x-3)(x-1) = 0$$
$$x = 3, \quad x = 1$$

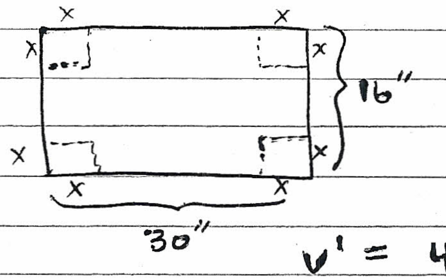
USE SECOND DERIVATIVE TEST TO VERIFY MAX DIMENSION:

$$V'' = -48 + 24x$$
$$V''|_{x=3} = -48 + 24(3) > 0 \Rightarrow \text{REL MINIMUM}$$
$$V''|_{x=1} = -48 + 24(1) < 0 \Rightarrow \text{REL } \underline{\text{MAXIMUM}}$$

MAX VOLUME WHEN $x = 1 \Rightarrow$ DIMENSIONS: $\boxed{4'' \times 4'' \times 1''}$

$$V = (6-2\cdot1)(6-2\cdot1)(1) = 16 \text{ in}^3$$

$\boxed{\#5}$



$\boxed{\text{PRIMARY EQUATION}}$

$$V = l\,w\,h$$

$$V = (30-2x)(16-2x)(x)$$

$$V' = 480 - 184x + 12x^2$$

$$12x^2 - 184x + 480 = 0$$

$$4\ (3x^2 - 46x + 120 = 0$$

$$4\ (3x-10)(x-12) = 0$$

$$x = 10/3\ ,\quad x = 12$$

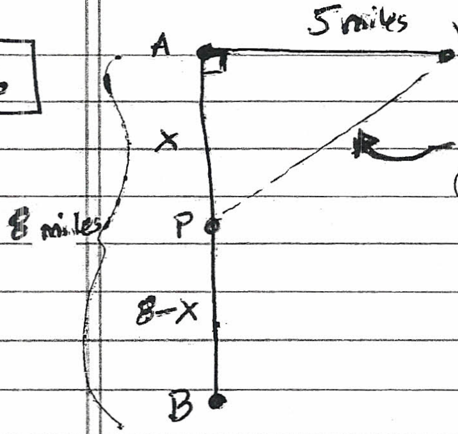
$$V'' = 24x - 184$$

$$V''\Big|_{x=10/3} = 24\left(\tfrac{10}{3}\right) - 184 < 0 \quad \therefore x = \tfrac{10}{3} \quad \text{Local MAX}$$

$$V''\Big|_{x=12} = 24(12) - 184 > 0 \quad \therefore x = 12 \quad \text{Local MIN}$$

Volume is MAXIMUM WHEN $X = 10/3''$

$\boxed{\#6}$



$x^2+5^2=c^2$

$\sqrt{x^2+25}=c$

$\boxed{\text{PRIMARY EQUATION}}$

COST: $C = \$100{,}000\left(\sqrt{x^2+25}\right) + \$75{,}000\,(8-x)$

$C' = 50{,}000\,(x^2+25)^{-1/2}(2x) - 75{,}000$

$C' = \dfrac{100{,}000\,x}{\sqrt{x^2+25}} - 75{,}000$

$\dfrac{100{,}000\,x}{\sqrt{x^2+25}} - 75{,}000 = 0$

$\dfrac{100{,}000\,x}{\sqrt{x^2+25}} = \dfrac{75{,}000}{1}$

$75{,}000\sqrt{x^2+25} = 100{,}000x$

$\sqrt{x^2+25} = \tfrac{4}{3}X$

$x^2+25 = \tfrac{16}{9}X^2$

$25 = \tfrac{16}{9}x^2 - x^2$

$25 = \tfrac{7}{9}x^2$

$x^2 = 25\left(\tfrac{9}{7}\right)$

$x^2 = \dfrac{225}{7}$

$x = \dfrac{15}{\sqrt{7}} \approx$ 5.67 MILES FROM POINT A

COST OF PIPE

FROM W to P to B ⇒ $100{,}000\sqrt{x^2+25} + 75{,}000(8-x) \approx \$930{,}700$

FROM W to B ⇒ $100{,}000\sqrt{5^2+8^2} \approx \$943.398$

FROM W to A to B ⇒ $100{,}000(5) + 75{,}000(8) \approx \$1{,}100{,}000$

#7

PRIMARY EQ

(MAXIMIZE PRODUCT)

SECONDARY EQ

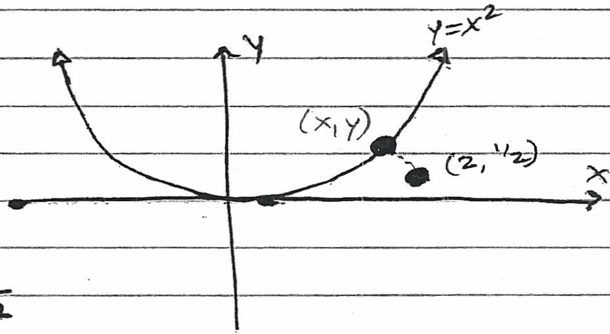
$$x + 2y = 108$$
$$x = 108 - 2y$$

$$P = xy$$
$$P = (108 - 2y)y$$
$$P = 108y - 2y^2$$
$$P' = 108 - 4y$$
$$108 - 4y = 0$$
$$4y = 108$$
$$\underline{y = 27} \Longrightarrow x = 108 - 2(27) = \underline{54}$$

27 and 54

#8

$$f(x) = x^2 \quad ; \quad (2, 1/2)$$



PRIMARY EQUATION

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$d = \sqrt{(x-2)^2 + (y - 1/2)^2}$$
$$d = \sqrt{(x-2)^2 + (x^2 - 1/2)^2}$$
$$d = \sqrt{\cancel{x^2} - 4x + 4 + x^4 - \cancel{x^2} + 1/4}$$
$$d = \sqrt{x^4 - 4x + 4\tfrac{1}{4}}$$
$$d' = \frac{4x^3 - 4}{2\sqrt{\cdots}}$$

(1, 1)

$$4x^3 - 4 = 0$$
$$x^3 = 1$$
$$\underline{x = 1} \Longrightarrow y = x^2$$
$$y = 1^2$$
$$\underline{y = 1}$$

#9

PRIMARY EQUATION

$$A = 2xy$$

$$A = 2x\left(\frac{400-4x}{3}\right)$$

$$A = \frac{800}{3}x - \frac{8}{3}x^2$$

$$A' = \frac{800}{3} - \frac{16}{3}x$$

$$\frac{800}{3} - \frac{16}{3}x = 0$$

$$\frac{800}{3} = \frac{16}{3}x$$

$$\left(\frac{3}{16}\right)\left(\frac{800}{3}\right) = x$$

$$x = 50'$$

SECONDARY EQ

$$400 = 4x + 3y$$

$$3y = 400 - 4x$$

$$y = \frac{400-4x}{3}$$

Diagrams ON WORKSHEET

$$50' \times \frac{200}{3}'$$

$$\Rightarrow y = \frac{400-4(50)}{3} = \frac{200'}{3}$$

#10

PRIMARY EQUATION

$$A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$A = x\left(\frac{32-2x-\pi x}{4}\right) + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$A = 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$A' = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x$$

$$32 - 4x - 2\pi x + \pi x = 0$$

$$x(-4 - 2\pi + \pi) = -32$$

$$x(-4 - \pi) = -32$$

$$x = \frac{32'}{4+\pi}$$

SECONDARY EQUATION

$$16 = x + 2y + \frac{1}{2}\pi x$$

$$32 = 2x + 4y + \pi x$$

$$\frac{32-2x-\pi x}{4} = y$$

$$\Rightarrow y = \frac{32 - 2\left(\frac{32}{4+\pi}\right) - \pi\left(\frac{32}{4+\pi}\right)}{4}$$

$$y = \frac{32(4+\pi) - 64 - 32\pi}{4(4+\pi)}$$

$$y = \frac{128-64}{4(4+\pi)}$$

$$y = \frac{16}{4+\pi}'$$

$$\frac{32'}{4+\pi} \text{ by } \frac{16'}{4+\pi}$$